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Existence of Solutions of Nonlinear Impulsive Differential Equations with Exponentially Dichotomous or Dichotomous Linear Part

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Sufficient conditions for the existence and uniqueness of solutions of nonlinear impulsive differential equations with exponentially dichotomous or dichotomous linear part are found.

1. INTRODUCTION

The present paper is devoted to the fundamental theory of impulsive differential equations in a Banach space. The main results are obtained under the assumption that the linear part of the impulsive differential equation considered is exponentially dichotomous or dichotomous. Some of the results obtained are new for equations without impulse effect as well.

2. STATEMENT OF THE PROBLEM

By I we shall denote $\mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$ and by J we shall denote either $\mathbb{Z}$ or $\mathbb{N} \cup \{0\}$.

Let X be an arbitrary Banach space with identity id_X. Consider the impulsive differential equation

$$\frac{dx}{dt} = A(t)x + F(t, x)\,\big|_{t \neq t_n} \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) + H_n(x(t_n)) \qquad (n \in J) \tag{2}$$

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where $T=\{t_n\}_{n\in J}$ is a finite or infinite sequence in I. We shall say that condition (H1) is satisfied if the following conditions hold:

H1.1. $A(t)$ $(t\in I)$ is a continuous operator-valued function with values in the Banach space $L(X)$ of all linear bounded operators acting in X.

H1.2. $Q_n\in L(X)$ $(n\in J)$.

H1.3. The function $F(t,x): I\times X\to X$ is continuous.

H1.4. $H_n: X\to X$ $(n\in J)$ are continuous operators.

H1.5.

$$t_n<t_{n+1} \qquad (n\in J) \tag{3}$$

$$\lim_{n\to\pm\infty} t_n=\pm\infty \tag{4}$$

Consider the linear impulsive differential equation

$$\frac{dx}{dt}=A(t)x \qquad (t\neq t_n, \quad n\in J) \tag{5}$$

$$x(t_n^+)=Q_n x(t_n) \qquad (n\in J) \tag{6}$$

As shown, for instance, in Zabreiko *et al.* (1988), if the operators Q_n have bounded inverse ones, then for the linear equation (5), (6) there exists a Cauchy operator $U(t)$ $(t\in I)$ by means of which each solution $x(t)$ of (5), (6) for which $x(\tau)=\xi\in X$ has the form

$$x(t)=U(t)U^{-1}(\tau)\xi \qquad (\tau, t\in I)$$

Definition 1. The linear impulsive differential equation (5), (6) is said to be *exponentially dichotomous* if in X there exist projectors P_+ and P_- $(P_+ + P_- = \mathrm{id}_X)$ and positive constants N_1, N_2, ν_1 and ν_2 for which

$$\|U(t)P_+U^{-1}(\tau)\|\le N_1 e^{-\nu_1(t-s)} \qquad (s\le t;\ t, s\in I) \tag{7}$$

$$\|U(t)P_-U^{-1}(\tau)\|\le N_2 e^{-\nu_2(s-t)} \qquad (t\le s;\ t, s\in I) \tag{8}$$

Definition 2. The linear impulsive differential equation (5), (6) is said to be *dichotomous* if in X there exist projectors P_+ and P_- $(P_+ + P_- = \mathrm{id}_X)$ and a positive constant D for which

$$\|U(t)P_+U^{-1}(\tau)\|\le D \qquad (s\le t; \quad t, s\in I) \tag{9}$$

$$\|U(t)P_-U^{-1}(\tau)\|\le D \qquad (t\le s; \quad t, s\in I) \tag{10}$$

The exponential dichotomy and the dichotomy of linear ordinary differential equations in a Banach space have been investigated, for instance, Daleckii and Krein (1974) and Massera and Schäffer (1966). Some initial investigations of these properties for linear impulsive differential equations have been carried out in Bainov *et al.* (1989*a,b*).

Definition 3. The nonnegative function $m(t)$ is said to be *integrally bounded* if for some $l > 0$ the following inequality is valid:

$$B(m(t)) = \sup_{t \in I} \int_t^{t+l} m(\tau)\, d\tau < \infty \tag{11}$$

Definition 4. The sequence of nonnegative numbers $\{m_n\}_{n \in J}$ is said to be *integrally bounded* if the following inequality is valid:

$$B(m_j) = \sup_{t \in I} \sum_{t \le t_j \le t+l} \|m_j\| < \infty \tag{12}$$

Set

$$L(m(t)) = \int_I m(\tau)\, d\tau \tag{13}$$

$$L(m_j) = \sum_{j \in J} m_j \tag{14}$$

We shall say that the function $F(t, x)$ $(t \in I, x \in X)$ and the operators H_j $(j \in J)$ satisfy the condition (H2) if the following conditions hold:
H2.1.

$$\|F(t, x)\| \le m(t) \qquad (\|x\| \le r, \quad t \in I) \tag{15}$$

H2.2.

$$\|F(t, x_1) - F(t, x_2)\| \le k(t)\|x_1 - x_2\| (\|x_1\|, \|x_2\| \le r, \quad t \in I) \tag{16}$$

H2.3.

$$\|H_j(x)\| \le m_j \qquad (\|x\| \le r, \quad j \in J) \tag{17}$$

H2.4.

$$\|H_j(x_1) - H_j(x_2)\| \le k_j\|x_1 - x_2\| \qquad (\|x\| \le r, \quad j \in J) \tag{18}$$

Definition 5. The function $F(t, x)$ and the operators H_j belong to the class $ED(a_1, a_2, a_3, a_4, r)$ if the functions $m(t)$ and $k(t)$ are integrally bounded and $B(m(t)) \le a_1$, $B(k(t)) \le a_2$, and the sequences $\{m_j\}$ and $\{k_j\}$ are integrally bounded and $B(m_j) \le a_3$, $B(k_j) \le a_4$.

Definition 6. The function $F(t, x)$ and the operators H_j are said to belong to the class $D(a_1, a_2, a_3, a_4, r)$ if the functions $m(t)$ and $k(t)$ are integrable on I and $L(m(t)) \le a_1$, $L(k(t)) \le a_2$, and the sequences $\{m_j\}$ and $\{k_j\}$ are summable on J and $L(m_j) \le a_3$, $L(k_j) \le a_4$.

3. MAIN RESULTS

Theorem 1. Let the following conditions hold:

1. The linear impulsive differential equation (5), (6) is exponentially dichotomous with projectors P_1 and P_2.
2. Conditions (H1) and (H2) hold.
3. $F(t, x)$ and H_j belong to the class $ED(a_1, a_2, a_3, a_4, r)$.
4. The operators Q_n have bounded inverse ones.

Then for an arbitrary $r>0$, for sufficiently small values of a_1, a_2, a_3, a_4 the impulsive equation (1), (2) has a unique solution $x(t)$ which is defined for $t\in\mathbb{R}$ and for which $\|x(t)\|\le r$ $(t\in\mathbb{R})$.

Proof. Let $C(X)$ be the space of all bounded functions $x\colon \mathbb{R}\to X$ which are continuous for $t\notin T$, have discontinuities of the first kind for $t\in T$, and are continuous from the left, with norm

$$|||x|||=\sup_{t\in\mathbb{R}}\|x(t)\|$$

The space $C(X)$ is Banach. Consider the operator $Q\colon C(X)\to C(X)$ defined by the formula

$$(Qx)(t)=\int_{-\infty}^{\infty} G(t,\tau)F(\tau,x(\tau))\,d\tau+\sum_{j=-\infty}^{\infty} G(t,t_j^+)H_j(x(t_j)) \tag{19}$$

where

$$G(t,\tau)=\begin{cases} U(t)P_1U^{-1}(\tau), & \tau<t \\ -U(t)P_2U^{-1}(\tau), & t<\tau \end{cases} \qquad (\tau, t\in\mathbb{R})$$

We shall show that the ball $|||x|||\le r$ is invariant with respect to Q and the operator Q is contracting. The following inequality is valid:

$$\begin{aligned}\|(Qx)(t)\| \le & \int_{-\infty}^{t} \|G(t,\tau)\|\,\|F(\tau,x(\tau))\|\,d\tau \\ & + \int_{t}^{\infty} \|G(t,\tau)\|\,\|F(\tau,x(\tau))\|\,d\tau \\ & + \sum_{t_j<t} \|G(t,t_j^+)\|\,\|H_j(x(t_j))\| \\ & + \sum_{t\le t_j} \|G(t,t_j^+)\|\,\|H_j(x(t_j))\| \end{aligned} \tag{20}$$

It is standard to verify that for the first and second integrals in (20) the following estimates are valid:

$$\int_{-\infty}^{t} \|G(t, \tau)\| \ \|F(\tau, x(\tau))\| \, d\tau \leq \frac{N_1 a_1}{1-e^{-\nu_1}}$$

$$\int_{t}^{\infty} \|G(t, \tau)\| \ \|F(\tau. x(\tau))\| \, d\tau \leq \frac{N_2 a_1}{1-e^{-\nu_2}}$$

Analogously for the sums in (20) we obtain the estimates

$$\sum_{t_j<t} \|G(t, t_j^+)\| \ \|H_j(x(t_j))\| \leq \frac{N_1 a_3}{1-e^{-\nu_1 l}}$$

$$\sum_{t\leq t_j} \|G(t, t_j^+)\| \ \|H_j(x(t_j))\| \leq \frac{N_2 a_3}{1-e^{-\nu_2 l}}$$

For sufficiently small a_1, a_3 the operator Q maps the ball $|||x|||\leq r$ into itself.

Let x_1, x_2 be arbitrary elements of the ball $|||x|||\leq r$. Then it is standard to verify that for $\|Qx_1(t)-Qx_2(t)\|$ the following estimate is valid:

$$\|(Qx_1)(t)-(Qx_2)(t)\|$$
$$\leq |||x_1-x_2|||\left(\frac{a_2N_1}{1-e^{\nu_1}}+\frac{a_2N_2}{1-e^{-\nu_2}}+\frac{a_4N_1}{1-e^{-\nu_1 l}}+\frac{a_4N_2}{1-e^{-\nu_2 l}}\right)$$

For sufficiently small a_2 and a_4 the operator Q is contracting in the ball $|||x|||\leq r$, hence it has a unique fixed point. It is not hard to verify that each solution of the impulsive equation (1), (2) lying in the ball $|||x|||\leq r$ is also a solution of the equation

$$x(t)=Qx(t)$$

and vice versa.

Theorem 1 is proved. ■

Remark 1. If the conditions of Theorem 1 are fulfilled and if, moreover, $F(t,0)=H_j(0)=0$ $(t\in\mathbb{R}, j\in\mathbb{Z})$, then $x=0$ is a unique solution of (1), (2) in $C(X)$.

Theorem 2. Let the following conditions hold:

1. The linear impulsive equation (5), (6) is dichotomous with projectors P_1 and P_2.
2. Conditions (H1) and (H2) hold.
3. $F(t, x)$ and H_j belong to the class $D(a_1, a_2, a_3, a_4, r)$.
4. The operators Q_n have bounded inverse ones.

Then for each r for sufficiently small values of a_1, a_2, a_3, and a_4 the impulsive equation (1), (2) has a unique solution $x(t)$ which is defined for $t \in \mathbb{R}$ and for which $\|x(t)\| \leq r$ $(t \in \mathbb{R})$.

Proof. In the proof of Theorem 1 it was mentioned that each solution $x(t)$ of the impulsive equation (5), (6) satisfies the equation

$$x(t) = \int_{-\infty}^{\infty} G(t, \tau) F(\tau, x(\tau))\, d\tau + \sum_{j=-\infty}^{\infty} G(t, t_j^+) H_j(x(t_j))$$

and vice versa. For $\|(Qx)(t)\|$, where the operator Q is defined in (19), we obtain the following estimate:

$$\begin{aligned}\|Qx(t)\| &\leq D \int_{-\infty}^{\infty} \|F(\tau, x(\tau))\|\, d\tau + D \sum_{j=-\infty}^{\infty} \|H_j(x(t_j))\| \\ &\leq D \int_{-\infty}^{\infty} m(\tau)\, d\tau + D \sum_{j=-\infty}^{\infty} m_j \\ &= D(L(m(t)) + L(m_j)) \leq D(a_1 + a_3)\end{aligned}$$

For sufficiently small a_1 and a_3, Q maps the ball $|||x||| \leq r$ into itself. Let x_1, x_2 be arbitrary elements of the ball $|||x||| \leq r$.

Then for $|||Qx_1 - Qx_2|||$ we obtain the estimate

$$|||Qx_1 - Qx_2||| \leq D|||x_1 - x_2|||(a_2 + a_4)$$

For sufficiently small a_2 and a_4 the operator Q is contracting and has just one fixed point in the ball $|||x||| \leq r$.

Theorem 2 is proved. ■

Theorem 3. Let the following conditions hold:

1. The linear impulsive equation (5), (6) is exponentially dichotomous.
2. Conditions (H1) and (H2) hold.
3. $F(t, x)$ and H_j belong to the class $ED(a_1, a_2, a_3, a_4, r)$.
4. The operators Q_n have bounded inverse ones.

Then for each r for sufficiently small a_1, a_2, a_3, and a_4 there exists $\rho < r$ such that for $\|\xi\| \leq \rho$ the impulsive equation (1), (2) has a unique solution $x(t)$ on $\mathbb{R}_+$ for which $P_1 x(0) = \xi$ and $\|x(t)\| \leq r$ $(t \in \mathbb{R}_+)$.

Proof. It is standard to check that each solution $x(t)$ of (1), (2) on $\mathbb{R}_+$ satisfies the equation

$$x(t) = (Qx)(t) \tag{21}$$

where Q is an operator defined by the formula

$$(Qx)(t) = U(t)\xi + \int_0^\infty G(t, \tau)F(\tau, x(\tau))\,d\tau + \sum_{j=0}^{\infty} G(t, t_j^+)H_j(x(t_j)) \tag{22}$$

and conversely, each solution of (21) lying in the ball $|||x||| \le r$ is a solution of (1), (2).

It is not hard to check that for $\|(Qx)(t)\|$ the following estimate is valid:

$$\|(Qx)(t)\| \le \frac{r}{2} + \left\{N_1 \int_0^t e^{-\nu_1(t-\tau)}\,\|F(\tau, x(\tau))\|\,d\tau + N_2 \int_t^\infty e^{-\nu_2(\tau-t)}\,\|F(\tau, x(\tau))\|\,d\tau\right\} + \left\{N_1 \sum_{t_j \le t} e^{-\nu_1(t-t_j)}\,\|H_j(x(t_j))\| + N_2 \sum_{t<t_j} e^{-\nu_2(t_j-t)}\|H_j(x(t_j))\|\right\}$$

For sufficiently small a_1 and a_3, Q maps the ball $|||x||| \le r$ into itself and it is a contracting operator.

Theorem 3 is proved. ■

Corollary 1. Let the conditions of Theorem 3 hold and let $x_1(t)$ and $x_2(t)$ be two solutions corresponding to the initial data ξ and η. Then the following estimate is valid:

$$\|x_1(t) - x_2(t)\| \le C_\mu e^{-\mu}\|\xi - \eta\| \qquad (t \in \mathbb{R}_+)$$

where $C_\mu > 0$ is a constant and $\mu \in (0, \min(\nu_1, \nu_2))$.

Proof. From representation (24) and the conditions of Theorem 3 for $u(t) = \|x_1(t) - x_2(t)\|$ there follows the validity of the inequality

$$u(t) \le \|\xi - \eta\|Ne^{-\nu t} + \int_0^t N_1 e^{-\nu(t-\tau)}m(\tau)u(\tau)\,d\tau + \int_t^\infty N_2 e^{-\nu(\tau-t)}m(\tau)u(\tau)\,d\tau + \sum_{t_j \le t} N_1 e^{-\nu(t-t_j)}m_j u(t_j) + \sum_{t_j > t} N_2 e^{-\nu(t_j-t)}m_j u(t_j)$$

which implies the assertion of Corollary 1 [see Theorem 3 of Bainov *et al.* (1989*a*)]. ■

Theorem 4. Let the following conditions hold:

1. The linear impulsive equation (5), (6) is dichotomous.
2. Conditions (H1) and (H2) hold.
3. $F(t, x)$ and H_j belong to the class $D(a_1, a_2, a_3, a_4, r)$.

Then for any $r > 0$ for sufficiently small values of a_1, a_2, a_3, and a_4 there exists $\rho < r$ such that for $\|\xi\| \leq \rho$ the impulsive equation (1), (2) has a unique solution $x(t)$ on $\mathbb{R}_+$ such that $P_1 x(0) = \xi$ and $\|x(t)\| \leq r$ $(t \in \mathbb{R}_+)$.

Proof. In the proof of Theorem 3 it was mentioned that each solution of (1), (2) lying in the ball $|||x||| \leq r$ satisfies equation (21) and vice versa. For $\|(Qx)(t)\|$, where the operator Q is defined in (22), we obtain the estimate

$$\|(Qx)(t)\| \leq D \cdot \frac{r}{2D} + D \int_0^\infty m(t)\, dt + D \sum_{j=0}^{\infty} m_j \leq \frac{r}{2} + D(a_1 + a_3)$$

If a_1 and a_3 are small enough, then $\|(Qx)(t)\| \leq r$.

Let x_1, x_2 be arbitrary elements of the ball $|||x||| \leq r$. Then the following inequality is valid:

$$|||Qx_1 - Qx_2||| \leq |||x_1 - x_2|||D(a_2 + a_4)$$

i.e., for sufficiently small a_2 and a_4 the operator Q is contracting.

Theorem 4 is proved. ■

Corollary 2. Let the conditions of Theorem 4 hold and let $x_1(t)$ and $x_2(t)$ be two solutions corresponding to the initial values ξ and η. Then for $D(a_1 + a_3) < 1$ the following estimate is valid:

$$\|x_1(t) - x_2(t)\| \leq \frac{D}{1 - D(a_1 + a_3)} \|\xi - \eta\| \qquad (t \in \mathbb{R}_+)$$

Proof. For $\|x_1(t) - x_2(t)\|$ the following estimate is valid:

$$\|x_1(t) - x_2(t)\| \leq \|\xi - \eta\| D + D \int_0^\infty k(\tau) \|x_1(\tau) - x_2(\tau)\|\, d\tau + D \sum_{j=0}^{\infty} k_j \|x_1(t_j) - x_2(t_j)\|$$

Set $u(t)=\|x_1(t)-x_2(t)\|$ and consider the equation

$$u(t)=\alpha+D\int_0^\infty k(\tau)u(\tau)\,d\tau+D\sum_{j=0}^\infty k_j u(t_j) \tag{23}$$

where $\alpha=\|\xi-\eta\|D$. Introduce the functional $\Phi\colon C\to\mathbb{R}_+$, where C is the space of all bounded on $\mathbb{R}_+$ functions with values in $\mathbb{R}_+$ which are continuous from the left, by the formula

$$(\Phi u)(t)=D\int_0^\infty k(\tau)u(\tau)\,d\tau+D\sum_{j=0}^\infty k_j u(t_j)$$

For the norm of Φ we obtain the estimate

$$\|\Phi\|\le D\left(\int_0^\infty k(\tau)\,d\tau+\sum_{j=0}^\infty k_j\right)\le D(a_1+a_3)$$

For sufficiently small a_1 and a_3 we have $\|\Phi\|<1$.

Let I_C be the identity of the space C. Then the equation $(I_C-\Phi)u=\alpha$ has a bounded solution $u(t)$, i.e., there exists a constant c for which $\|u(t)\|\le c$ $(t\in\mathbb{R}_+)$. We shall estimate the constant c from equation (26):

$$c\le\alpha+Dc\int_0^\infty k(\tau)\,d\tau+Dc\sum_{j=0}^\infty k_j\le\alpha+Dca_1+Dca_3$$

i.e.,

$$c\le\frac{\alpha}{1-D(a_1+a_3)}$$

Finally we obtain

$$\|x_1(t)-x_2(t)\|\le\frac{\|\xi-\eta\|D}{1-D(a_1+a_3)}$$

Corollary 2 is proved. ■

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